Puzzles and Mathematics.

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There are many puzzles with a mathematical background. They can and should be used to train mathematical thinking and to make people aware of mathematical structures.

1. Let us start with a look at some examples, say Tangram or the Soma Cube. Here, several quite chaotic pieces have to be assembled to create an object with complete regularity (a square, a cube). Phrasing it differently, a nice object is dissected into smaller pieces in order to destroy all its symmetries and one has to recover the original object.

Dissections have always been a splendid method for creating puzzles, the most astonishing one may be the square cut of a tetrahedron. Here, we deal with a puzzle which consists just of two pieces, but which (for psychological reasons) is irritating for a neophyte: what is misleading is the symmetry of the small pieces. Similarly, one may cut a cube into two halves such that the cut is a regular hexagon. For a given cube, there are four such hexagons, and their convex hull is the cuboctahedron, thus a model of the root system A_3.

Some very interesting puzzles use several identical pieces which have to be combined. For example, starting again with a cube, its center yields with any of its faces a square pyramid. These six pyramids may be decorated in various ways in order to form a star or a ball: we obtain in this way objects which fall apart into six identical pieces. Note that the six central pyramids show the diagonals of the cube and many puzzles designed by Stewart Coffin focus the attention to these diagonals.

The Platonic solids (and, more generally, the Archimedean ones) have always been a fruitful source for inspiration. An icosahedron hides a very important secret: first of all, any edge and its opposite yield a rectangle which turns out to be a golden rectangle (one of the most prominent way to obtain golden rectangles). The 30 edges provide 15 rectangles, and we obtain in this way 5 triples of pairwise orthogonal rectangles. One should observe that such a triple exhibits the coordinate planes of a coordinate system. The coordinate planes of a coordinate system are also the basis of puzzles which ask to get this configuration starting with planar pieces.

The so-called Boss Puzzle (or 15-Puzzle) consists of a frame filled with square tiles numbered from 1 to 15 which have to be rearranged into order. Thus we deal with permutations. The important observation is that only even permutation can be realized, thus not all arrangements of the tiles can be achieved. Many other puzzles, such as the Rubik Cube (and numerous variations) are based on the use of permutations and the permutation groups which arise in this way may be very complicated.

Many puzzles are made of cubes of equal size which are put together face by face. But there are similar puzzles using spheres of equal size, thus we deal with sphere packings. For long time, the problem of the densest sphere packing was open. In 1831, Gauss had determined the densest lattice packings, but it took until 2014 that Hales finished his proof that the lattice packings are actually the densest possible packings at all (as conjectured by Kepler already in 1611). One should be aware that there are densest sphere packings which are aperiodic. If one starts as a first layer with a plane, densely packed with spheres, If one starts as a first layer with a plane, densely packed with spheres, one may put a similar layer on top of it, so that any sphere of the second layer touches three of the first one. Note that this can be done in two ways. Thus looking at three layers, we actually obtain essentially different configurations. In this way, one can construct both periodic as well as aperiodic dense sphere packings of the space.

Let me end this short list of examples with may-be the simplest puzzling object: Starting with a sheet of paper, one can obtain with two scissor cuts and a single folding line a three-dimensional object which, being put on the desk, may confuse any visitor for a while.

2. Puzzles were popular already in the 18th century, as parlor games, but also as tools for instruction. And often there was a cryptic connotation to Chinese tradition (say the label Chinese rings, or Tangram). In 1893, an important reference book was published: Hoffmann's "Puzzles - Old and New". In the second half of the 20th century, the books by Pieter van Delft and Jack Botermans were a great source of inspiration.

At the beginning of the 20th century, the Anker puzzles (variations of Tangram) were very popular in many countries. During the first world war, they were played in the dugouts of the different enemy warriors.

In the hype of the Rubik cube (invented by E. Rubik in 1974), the Nederlandse Kubus Club started a journal CFF (Cubism For Fun) and Jerry Slocum gathered puzzle designers and puzzle freaks at the yearly International Puzzle Party IPP which usually rotates between the US, Japan and Europe.

One has to praise Martin Gardner highly for popularizing mathematical ideas, spreading information about new developments in mathematics and showing their relevance in daily life. His columns in *Scientific American* were labeled “Mathematical Games” and they featured a lot of puzzles.

3. The analysis of the mathematical background of puzzles has to be seen in the broader context of general mathematical awareness: the daily life provides a lot of opportunities for spotting mathematical structures in art and nature.

All the basic ingredients of music (the harmonies and pitches, the rhythm) are mathematical. One should be aware that music is really the realm of fractions. This concerns what one may call the local structure of music. In addition, also the global structure may show mathematical characteristics, as demonstrated by the use of symmetries (see for example Bach), and of aleatoric features (Mozart, Schönberg, Arvo Pärt).

In nearly all cultures, there has been a rich tradition to use ornaments for architecture, for pottery, furniture, textiles or wallpapers, often based on geometrical shapes and patterns. The tesselations used in Islamic art provide examples for all the 17 plane crystallographic groups.

The Italian Renaissance painters and architects studied linear perspective and used it in their drawings. Leonardo da Vinci and Dürer stressed the relevance of mathematical structures in art. The architecture of many periods (from the classical Greek and Roman times via the Romanesque period and the Neoclassicism to Bauhaus and the International Style) is based on using cuboids. But also hyperbolas and other conic sections have often attracted the interest of architects. And we should mention the geodesic domes of Bauersfeld and Buckminster Fuller, as well as the tensegrities of Fuller and Snelson.

In the early 20th century, artists started to experiment with abstract structured compositions. The geometric constructivist art often uses just right-angled shapes and only few clear colors. On the other hand, there are the optical illusions: the Reutersvärd triangle, the Penrose triangle, and other “impossible objects” (usually based on our desire to interpret two-dimensional drawings as three-dimensional objects). Escher tried to explore infinity, using for example hyperbolic geometry as well as tessellations.

But there are also many interesting features of elementary mathematics in nature itself, pinpointed for example in photos by Blossfeldt, or in drawings by Haeckel; they are a good source for inspiration. It is well-known that the theory of fractals provides a surprising way to obtain chaotic, yet well-arranged behavior. Fractals are encountered ubiquitously in nature.

4. But a warning is necessary. One often may be inclined to see some mathematical rules in daily life, in nature and art, which after all turn out to be just illusionary: People are trained to look for patterns and to attach to them some meaning.

There is an entertaining film with the title *Donald in Mathmagic Land* featuring Donald Duck. It was produced by Walt Disney (and even nominated for an Academy Award) and outlines connections first between mathematics and music, and then moves to architecture and arts. It claims that the golden rectangle appears in many ancient buildings, such as the Parthenon and the Notre Dame cathedral, that famous paintings and sculptures such as the Mona Lisa and the Venus de Milo contain several golden rectangles and that also modern buildings such as the United Nations building in New York City show the use of the golden rectangle.

The idea behind the film (and behind many corresponding publications) goes back to Adolf Zeising, a German psychologist from the 19th century with some interest in mathematics who claimed to have found the golden ratio nearly everywhere in nature and in art, in particular in the proportions of the human body. For him, the appearance of the golden ratio was a kind of universal law, a hidden secret used already in ancient times. But the selection of proportions which he chose for highlighting the golden ratio as an important value are really arbitrary, and even in his main examples the exact measures deviate well from the golden ratio. Also, no empirical evidence that people rate golden rectangles as more beautiful than others has been found. It has to be amitted that Le Corbusier liked the ideas of Zeising, and he was involved in the design of the United Nations building. But he may be one of the few architects who actually used golden rectangles in a prominent way.

If there is any dominance of proportions similar to the golden ratio, then it may be related to more complicated patterns described by the sequence of the Fibonacci numbers.

5. A lot of topics in elementary mathematics can be illustrated by puzzles. Elementary mathematics concerns numbers and shapes: the general framework of the world we live in. What always matters are the symmetries which occur (some may be hidden, but still are essential), this is the basis of the order in nature and then also in art.

Numbers are used for counting, but usually not a single number matters, but the number in relation to other numbers. The relation between two numbers can be expressed in many different ways, very important is often their proportion, the corresponding fraction. Note that fractions such as

7 : 9 and 8 : 8 with nearly equal value, give rise to what one may call *vanishing* puzzles: that the rearrangement of pieces which apparently fill some space allows to insert an additional piece.

The relevance of proportions can be seen already in nature: The appearance of Fibonacci numbers in various contexts is amazing. We have mentioned already that, in music, harmonies are described by fractions, and, similarly, fractions play a decisive role in art and architecture.

There are block puzzles by Slothouber-Graatsma and Conway which are based on parity. To find the solution, one has to realize that the number of cubes in any slice is odd, whereas most blocks will contribute an even number of cubes.

Sloane's On-line Encyclopedia of integer sequences is always an inspiring source of information. Note that here one deals with sequences (indexed, as usual, by natural numbers), but also with number triangles. Let us mention some sequences of numbers which are important in our context. Exponential sequences such as 2^n describe the increase of difficulties say when looking at the Tower of Hanoi or the Chinese Ring Puzzle. The factorial numbers count the number of permutations, thus yield the cardinality of the symmetric groups, the subfactorial numbers count the number of fixpoint-free permutations. There are the Fibonacci numbers, which are found in many growth problems in nature, and the Catalan numbers which occur when counting paths in a grid. Here, we are in the realm of combinatorics.

Now let us draw the attention to shapes, thus to topology and geometry. As one of the first examples of a mathematical problem used for recreation one often mentions the Seven Bridges of Königsberg. Actually, it inspired Euler in 1735 to lay the foundations of graph theory, this may be considered as the beginning of topology. There are also other concepts of graph theory, such as planarity of graphs and chromatic numbers which can be discussed without further pre-knowledge. Dealing with polyeders, it is the Euler characteristic which plays a decisive role: for example, it controls the number of pentagons in a geodesic dome.

The common target of the string puzzles is to remove a closed string from a somewhat complicated wooden or wire construction. It looks apparently impossible, since one has the impression that one deals with two linked circles. One should be aware that the linkage of circles is an important feature in topology (and knot theory), it is for example the basis of stable homotopy theory. In this context, we have to mention the Borromean rings, these are three linked circles so that after removing any one of them, the remaining two are no longer linked. In contrast, one obtains interesting links of cycles by dealing with curves on a torus. In particular the Villarceau circles are pairwise linked; they are obtained by suitable cuts of the torus with a plane.

There is the local geometry of the real plane, with symmetries given by rotations and reflections, but also the global geometry of the real plane, using in addition translations, thus dealing, for example, with tesselations. Then there is the 3-dimensional geometry which takes care of all the polyhedra mentioned already, of spheres and the tori. And we should also glance at projections of objects which live in the 4-dimensional space, such as the Kleinian bottle. Finally, for many considerations, for example for dealing with fractals, one has to use complex numbers and complex geometry. Already for a proper understanding of the geometry of the real plane, one should interpret the plane as the set of complex numbers. And the Villarceau circles should be considered as subsets of the complex plane.

In some older presentation, I have stressed that *algebra is geometry is algebra*: that it is important to visualize algebraic concepts by using geometric ideas, and, conversely, to rewrite geometrical problems using algebraic formulae. This relationship between algebra and geometry is already implicitly used by Euclid and seems to describe some of the Pythagorean ideas. There are several planar wood puzzles which illuminate proofs of the Pythagorean theorem. The most impressive relation between geometrical data and algebra seem to be given by the root systems which were introduced in Lie theory in order to describe for example semi-simple algebraic groups. Note that the low dimensional root systems yield basic configurations in the plane and in the space.

6. It should be worthwhile to insert a short section on the role of mathematics in general. Centuries ago, when philosophy challenged theology as supreme scholarship, the old concept to consider philosophy as a servant of theology was interpreted by Kant by saying that philosophy should not be the servant who carries the tail of the dress of the lady, but the torch to show her the way. In the meantime, the role of the faculties has been shifted, theology no longer plays a role; it is now mathematics who challenges the role of philosophy: mathematics will not carry the tail of the dress of the lady, but the torch to show her the way.

7. Mathematical education makes students aware of mathematical structures in order to enable them to understand their environment, to compare and to predict. Puzzles may be used as an ideal tool.

First of all, puzzles are an ideal tool for training to solve problem. A first attempt for finding a solution will always be trial and error, but a systematic approach is of course desirable. Using puzzles, one has the big advantage to work with variations, to increase difficulties step by step, in a controlled way. On the other hand, two puzzles may look rather similar, but may need completely different strategies to be solved. In this way, working with puzzles will promote lateral thinking.

Some puzzles seem to be impossible to solve, there may be some tricky move which one tends to overlook. Puzzles can serve as a a good source for discussing aspects of impossibility: that apparently impossible tasks may be achieved quite easily, whereas there are other problems where definitely no solution exist. Working with puzzles will help to questions the first intuition. In this way, one becomes aware that there do exist a lot of counter-intuitive examples.

In former time, an essential part of mathematical education was Euclidean geometry, and such a syllabus meant that students learn in a rigorous way what a mathematical proof is. It is a disaster if school education attaches less attention to proofs, since they are the heart of mathematical thinking. Actually, the use of puzzles may counterbalance this development. To phrase it differently: The study of mathematical problems is the only place where students become aware of strict impossibility and there are several puzzles (for example the Boss Puzzle) which show that some target may be impossible to achieve, and that there is a rigorous proof about the impossibility.

Many puzzles are good tools for training the 3-dimensional perception, and indeed, some of them, for example Piet Hein's Soma cube, were invented just for this reason.

The use of puzzles in school education should brighten up the teaching of mathematics. After all, mathematical training is hard work in order to achieve sufficient practice. What are needed in addition are surprises, since they foster the memory. To insert mathematical puzzles may be helpful to provide challenges in a relaxed environment. Mathematical training is hard work, but should be also a lot of fun.

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This text was compiled for the Xiamen puzzle collection. It is obvious that it is written from a European perspective, without references to Chinese perceptions, Chinese history, Chinese philosophy. Thus it may need a corresponding augmentation and amplification by a Chinese scholar.

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